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A crystal theoretic method for finding rigged configurations from paths

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Abstract

The Kerov–Kirillov–Reshetikhin (KKR) bijection gives one-to-one correspondences between the set of highest paths and the set of rigged configurations. In this paper, we give a crystal theoretic reformulation of the KKR map from the paths to rigged configurations, using the combinatorial R and energy functions. This formalism provides a tool for analysis of the periodic box-ball systems.

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1. Introduction

The Kerov–Kirillov–Reshetikhin (KKR) bijection [1–3] gives the combinatorial one-to-one correspondences between the set of highest weight elements of tensor products of crystals [4, 5] (which we call the highest paths) and the set of combinatorial objects, called the rigged configurations. This bijection was originally introduced as an essential tool to derive a new expression (called fermionic formulae) of the celebrated Kostka–Foulkas polynomials [6]. The background of this expression is the Bethe ansatz for the Heisenberg spin chain [7] and, in this context, the rigged configurations form an index set of the eigenvalues and eigenvectors of the Hamiltonian. To date, the fermionic formulae have been extended to a wider class of representations and proved in several cases (see, e.g., [8–12] for the current status of the study).

Recently, the KKR bijection itself became an active subject of the study. The fundamental observation [13] is that the KKR bijection is an inverse scattering transform of the box-ball systems, which is the prototypical example of the ultradiscrete soliton systems introduced by Takahashi–Satsuma [14, 15]. In this context, the rigged configurations are regarded as action and angle variables for the box-ball systems. This observation leads to derivation [16, 17] of general solutions for the box-ball systems for the first time.

Therefore, it is natural to ask what the representation theoretic origin of the KKR bijection is. A partial answer was given in the previous paper [16], and it is substantially used in the derivation in [17]. However, the formalism in [16] works only for the map from the rigged configuration to paths, and an extension of the formalism to the inverse direction seems to have essential obstructions. Up to now, crystal interpretation for the map from the paths to rigged configurations (which we denote by ϕ) remains open. A closely related problem is what the representation theoretic origin of the mysterious combinatorial algorithm of the original definition of ϕ is. In fact, the formalism in [16] gives an alternative representation theoretic map for ϕ^{-1} while it does not give meanings to the combinatorial procedures such as vacancy numbers or singular rows. We remark that in section 2.7 of [13], there is decomposition of the $\mathfrak{sl}_n$ type ϕ into successive computation of the $\mathfrak{sl}_2$ type ϕ. However, it finally uses a combinatorial version of ϕ; hence, it is not a complete crystal interpretation of ϕ.

One of the main aims of the present paper is to give a crystal interpretation for the $\mathfrak{sl}_2$ type ϕ by clarifying the representation theoretic origin of the original combinatorial procedure of ϕ (see theorem 3.3). In our formalism, the combinatorial procedure of ϕ is identified with differences of energy functions called local energy distribution and indeed we can read off all information about the rigged configurations from them. In terms of the box-ball systems, the local energy distributions clarify which letters of a given path correspond to which soliton even if they are in a multiply scattering state.

Another aim of the paper is to provide a tool for analysis of the periodic box-ball systems [18, 19]. In our $\mathfrak{sl}_2$ case formalism, there is a remarkably nice property (see proposition 4.4). Namely, the solitons which appeared in the local energy distribution are always separated from each other. This leads to an alternative version of our procedure as given in theorem 3.6. The importance of this reformulation is that when we apply the formalism to find the action and angle variables of the periodic box-ball systems [20], we do not need to cut paths and treat them as non-periodic paths. This improves the inverse scattering formalism of [20] and theta function formulae of [21, 22] in a sense that we treat paths genuinely as periodic. As a byproduct, we give an intuitive picture of the basic operators which are the key to define angle variables in [20] (see remark 3.7). We remark that there is another approach to the initial value problem of the periodic box-ball systems [23]. Although their combinatorial method and our representation theoretic method are largely different, it will be important to clarify the relationship between these two approaches.

The present paper is organized as follows. In section 2, we review the combinatorial R and energy functions following [24]. In section 3, we formulate our main results (theorems 3.3 and 3.6). In section 4, we give proof of these theorems. Section 5 is the summary. In appendix A, we recall the KKR bijection and collect necessary facts, and in appendix B, we collect necessary facts about the time evolution operators T_l.

2. Combinatorial *R* and energy functions

In this section, we introduce necessary facts from the crystal bases theory, namely the combinatorial R and energy functions. Let B_k be the crystal of k-fold symmetric powers of the vector (or natural) representation of $U_q(\mathfrak{sl}_2)$. As the set, it is

$$B_k = \left\{(x_1, x_2) \in \mathbb{Z}^2_{\geqslant 0} \,\middle|\, x_1 + x_2 = k\right\}. \tag{1}$$

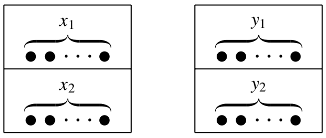
We usually identify elements of B_k as the semi-standard Young tableaux:

$$(x_1, x_2) = \boxed{\overbrace{1\cdots 1}^{x_1}\overbrace{2\cdots 2}^{x_2}}, \tag{2}$$

i.e. the number of letters i contained in a tableau is x_i.

For two crystals B_k and B_l of $U_q(\mathfrak{sl}_2)$, one can define the tensor product $B_k \otimes B_l = \{b \otimes b' \mid b \in B_k, b' \in B_l\}$. Then we have a unique isomorphism $R : B_k \otimes B_l \xrightarrow{\sim} B_l \otimes B_k$, i.e. a unique map which commutes with actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$. We call this map combinatorial R and usually write the map R simply by $\simeq$.

In calculation of the combinatorial R, it is convenient to use the diagrammatic technique due to Nakayashiki–Yamada (Rule 3.11 of [24]). Consider the two elements $x = (x_1, x_2) \in B_k$ and $y = (y_1, y_2) \in B_l$ respectively. Then we draw the following diagram to express the tensor product $x \otimes y$:

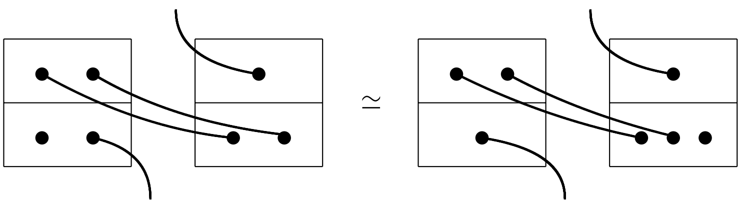


The combinatorial R matrix and energy function H for $x \otimes y \in B_k \otimes B_l$ (with $k \geqslant l$) are calculated by the following rule.

(1) Pick any dot, say $\bullet_a$, in the right column and connect it with a dot $\bullet'_a$ in the left column by a line. The partner $\bullet'_a$ is chosen from the dots whose positions are higher than those of $\bullet_a$. If there is no such dot, we return to the bottom and the partner $\bullet'_a$ is chosen from the dots in the lower row. In the former case, we call such a pair 'unwinding' and, in the latter case, we call it 'winding'.
(2) Repeat procedure (1) for the remaining unconnected dots $(l - 1)$ times.
(3) The action of the combinatorial R matrix is obtained by moving all unpaired dots in the left column to the right horizontally. We do not touch the paired dots during this move.
(4) The energy function H is given by the number of unwinding pairs.

The number of winding (or unwinding) pairs is sometimes called the winding (or unwinding, respectively) number of tensor product. It is known that the resulting combinatorial R matrix and the energy functions are not affected by the order of making pairs [24, propositions 3.15 and 3.17]. In the above description, we only consider the case $k \geqslant l$. The other case $k \leqslant l$ can be done by reversing the above procedure, noting the fact $R^2 = \mathrm{id}$. For more properties, including that the above definition indeed satisfies the axiom, see [24].

Example 2.1. Corresponding to the tensor product $\boxed{1122} \otimes \boxed{122}$, we draw the diagram given on the left-hand side of



By moving one unpaired dot to the right, we obtain

$$\boxed{1122} \otimes \boxed{122} \simeq \boxed{112} \otimes \boxed{1222}. \tag{3}$$

Since we have two unwinding pairs, the energy function is $H\left(\boxed{1122} \otimes \boxed{122}\right) = 2$.

Consider the affinization of the crystal B. As the set, it is

$$\mathrm{Aff}(B) = \{b[d] \mid b \in B, d \in \mathbb{Z}\}. \tag{4}$$

Integers d of $b[d]$ are called modes. For the tensor product $b_1[d_1] \otimes b_2[d_2] \in \mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l)$, we can lift the above definition of the combinatorial R as follows:

$$b_1[d_1] \otimes b_2[d_2] \overset{R}{\simeq} b'_2[d_2 - H(b_1 \otimes b_2)] \otimes b'_1[d_1 + H(b_1 \otimes b_2)], \tag{5}$$

where $b_1 \otimes b_2 \simeq b'_2 \otimes b'_1$ is the combinatorial R defined in the above.

Remark 2.2. The piecewise linear formula to obtain the combinatorial R and the energy function is also available [25]. This is suitable for computer implementation. For the affine combinatorial $R : x[d] \otimes y[e] \simeq \tilde{y}[e - H(x \otimes y)] \otimes \tilde{x}[d + H(x \otimes y)]$, we have

$$\begin{aligned} &\tilde{x}_i = x_i + Q_i(x, y) - Q_{i-1}(x, y), \qquad \tilde{y}_i = y_i + Q_{i-1}(x, y) - Q_i(x, y), \\ &H(x \otimes y) = Q_0(x, y), \\ &Q_i(x, y) = \min(x_{i+1}, y_i), \end{aligned} \tag{6}$$

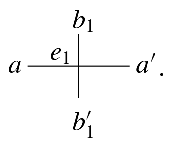
where we have expressed $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$. All indices i should be considered as $i \in \mathbb{Z}/2\mathbb{Z}$.

3. Local energy distribution and the KKR bijection

In this section, we reformulate the combinatorial procedure ϕ in terms of the energy functions of crystal base theory. See appendix A for explanation of ϕ. In order to do this, it is convenient to express actions of the combinatorial R by vertex-type diagrams. First, we express the isomorphism of the combinatorial R matrix:

$$a \otimes b_1 \simeq b'_1 \otimes a' \tag{7}$$

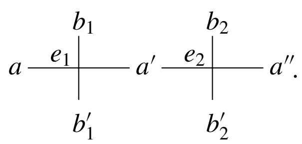
and the corresponding value of the energy function $e_1 := H(a \otimes b_1)$ by the following vertex diagram:



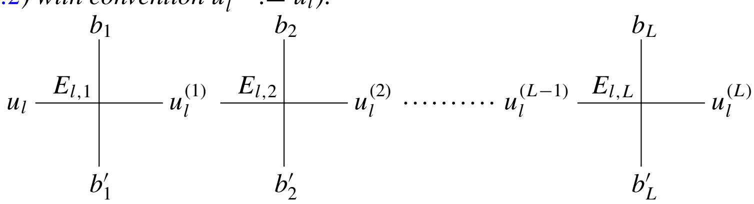
If we apply combinatorial R successively as

$$a \otimes b_1 \otimes b_2 \simeq b'_1 \otimes a' \otimes b_2 \simeq b'_1 \otimes b'_2 \otimes a'', \tag{8}$$

with the energy function $e_2 := H(a' \otimes b_2)$, then we express this by joining two vertices as follows:



Definition 3.1. *For a given path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, we define local energy $E_{l,j}$ by $E_{l,j} := H\big(u_l^{(j-1)} \otimes b_j\big)$. Here, in the diagrammatic expression, $u_l^{(j-1)}$ are defined as follows (see also (B.2) with convention $u_l^{(0)} := u_l$):*



Here, we denote $T_l(b) = b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$. *We define* $E_{0,j} = 0$ *for all* $1 \leqslant j \leqslant L$. *We also use the following notation:*

$$E_l := \sum_{j=1}^{L} E_{l,j}. \tag{9}$$

In other words, $u_l[0] \otimes b \overset{R}{\simeq} T_l(b) \otimes u_l^{(L)}[E_l]$, *where we have omitted modes for* b *and* $T_l(b)$.

For a given path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ $\left(b_i \in B_{\lambda_i}\right)$, we create a path $b' = b \otimes \boxed{1}^{\otimes\Lambda}$, where $\Lambda > \lambda_1 + \cdots + \lambda_L$. Then we always have $u_l^{(L+\Lambda)} = u_l$ for arbitrary l (proposition B.1 (1)). Under such a circumstance, it is known that the sum E_l is conserved quantities of the box-ball system; $E_l(T_k(b')) = E_l(b')$. The proof is based on successive application of the Yang–Baxter equation (see theorem 3.2 of [26] and section 3.4 of [25]). However, for our purpose, we need more detailed information such as $E_{l,j}$.

Lemma 3.2. *For a given path* $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, *we have* $E_{l,j} - E_{l-1,j} = 0$ *or 1, for all* $l > 0$ *and for all* $1 \leqslant j \leqslant L$.

Proof. First we give a proof when $l = 1$, i.e. we show $E_{1,j} - E_{0,j} = 0$ or 1. In this case, we have $u_l, u_l^{(i)} \in B_1$ and $E_{0,j} = 0$. Since $H(x \otimes y) = 0$ or 1 for all $x \in B_1$ and all $y \in B_k$, the proof follows.

Now we consider possible values for $E_{l,j} - E_{l-1,j}$. In order to do this, we show that the difference between tableaux representations of $u_l^{(j)}$ and $u_{l-1}^{(j)}$ is only one letter. More precisely, we show that if $u_{l-1}^{(j)} = (x_1, x_2)$, then $u_l^{(j)} = (x_1 + 1, x_2)$ or $u_l^{(j)} = (x_1, x_2 + 1)$. We show this claim by induction on j. For the $j = 0$ case, it is true because $u_{l-1}^{(0)} = u_{l-1} = (l - 1, 0)$ and $u_l^{(0)} = u_l = (l, 0)$, by definition. Suppose that the above claim holds for all $j < k$ for some k. In order to compare $u_{l-1}^{(k)}$ and $u_l^{(k)}$, consider the isomorphisms $u_{l-1}^{(k-1)} \otimes b_k \simeq b'_{l-1,k} \otimes u_{l-1}^{(k)}$ and $u_l^{(k-1)} \otimes b_k \simeq b'_{l,k} \otimes u_l^{(k)}$ respectively. By assumption, the difference between $u_{l-1}^{(k-1)}$ and $u_l^{(k-1)}$ is one letter. Recall that in calculating the combinatorial R, the order of making pairs can be chosen arbitrary. Therefore, in $u_l^{(k-1)} \otimes b_k$, first we can make all pairs that appear in $u_{l-1}^{(k-1)} \otimes b_k$, and next we make the remaining one pair. This means the difference of number of unwinding pairs, i.e. $E_{l,k} - E_{l-1,k}$, is 0 or 1. To make the induction proceed, note that this fact means that the difference between $u_{l-1}^{(k)}$ and $u_l^{(k)}$ is also one letter. □

The following theorem gives crystal theoretic reformulation of the KKR map ϕ. See appendix A for explanation of the unrestricted rigged configurations.

Theorem 3.3. *Let* $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}$ *be an arbitrary path.* b *can be the highest weight or non-highest weight. Set* $N = E_1(b)$. *We determine the pair of numbers* $(\mu_1, r_1), (\mu_2, r_2), \ldots, (\mu_N, r_N)$ *by the following procedure from step 1 to step 4. Then the resulting* $(\lambda, (\mu, r))$ *coincides with the (unrestricted) rigged configuration* $\phi(b)$.

(1) Draw a table containing $(E_{l,j} - E_{l-1,j} = 0, 1)$ *at the position* (l, j), *i.e. at the lth row and the jth column. We call this table local energy distribution.*

(2) Starting from the rightmost 1 *in the* $l = 1$*st row, pick one* 1 *from each successive row. The* 1 *in the* $(l + 1)$*th row must be weakly right of the* 1 *selected in the lth row. If there is no such* 1 *in the* $(l + 1)$*th row, the position of the lastly picked* 1 *is called* (μ_1, j_1). *Change all selected* 1 *into* 0.

(3) Repeat step 2 for $(N-1)$ times to further determine $(\mu_2, j_2), \ldots, (\mu_N, j_N)$, thereby making all 1 *into* 0.

(4) Determine $r_1, \ldots, r_N$ by

$$r_k = \sum_{i=1}^{j_k-1} \min(\mu_k, \lambda_i) + E_{\mu_k, j_k} - 2\sum_{i=1}^{j_k} E_{\mu_k, i}. \tag{10}$$

The proof of theorem 3.3 will be given in the following section. As we will see in proposition 4.4, the groups obtained in the above theorem have no crossing with each other. Therefore, when we search 1 of the $(l+1)$th row in step 2, we have at most one candidate, i.e. we can uniquely determine such 1.

Example 3.4. For example of theorem 3.3, we consider the following path:

$$b = \boxed{1111} \otimes \boxed{11} \otimes \boxed{2} \otimes \boxed{1122} \otimes \boxed{1222} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{22} \tag{11}$$

Corresponding to step 1, the local energy distribution is given by the following table (j stands for the column coordinate of the table):

	1111	11	2	1122	1222	1	2	22
$E_{1,j} - E_{0,j}$	0	0	1	0	1	0	1	0
$E_{2,j} - E_{1,j}$	0	0	0	1	0	0	0	1
$E_{3,j} - E_{2,j}$	0	0	0	1	0	0	0	0
$E_{4,j} - E_{3,j}$	0	0	0	0	1	0	0	0
$E_{5,j} - E_{4,j}$	0	0	0	0	1	0	0	0
$E_{6,j} - E_{5,j}$	0	0	0	0	0	0	0	1
$E_{7,j} - E_{6,j}$	0	0	0	0	0	0	0	0

Following steps 2 and 3, letters 1 contained in the above table are found to be classified into three groups, as indicated in the following table:

	1111	11	2	1122	1222	1	2	22
$E_{1,j} - E_{0,j}$			3		2*		1	
$E_{2,j} - E_{1,j}$				3				1*
$E_{3,j} - E_{2,j}$				3				
$E_{4,j} - E_{3,j}$					3			
$E_{5,j} - E_{4,j}$					3			
$E_{6,j} - E_{5,j}$								3*
$E_{7,j} - E_{6,j}$								

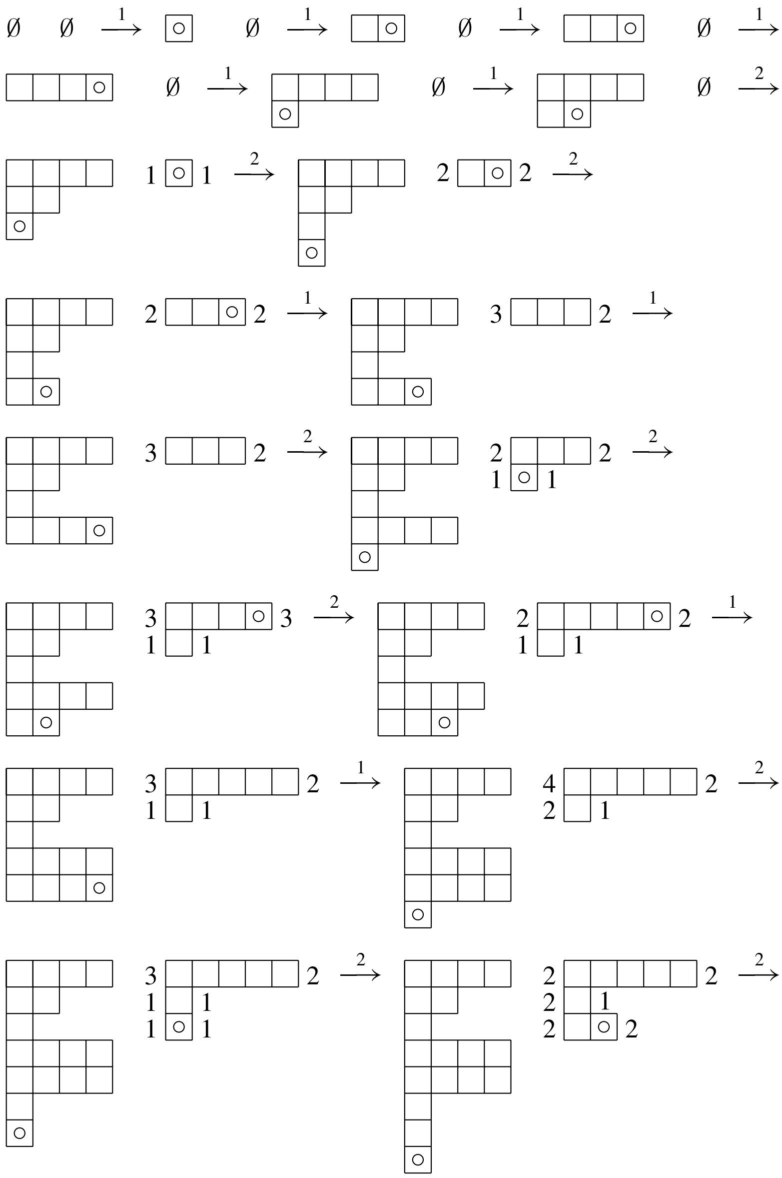
From the above table, we see that the cardinalities of groups 1, 2 and 3 are 2, 1 and 6, respectively. Also, in the above table, positions of $(\mu_1, j_1), (\mu_2, j_2)$ and (μ_3, j_3) are indicated by asterisks. Their explicit locations are $(\mu_1, j_1) = (2, 8), (\mu_2, j_2) = (1, 5)$ and $(\mu_3, j_3) = (6, 8)$ respectively.

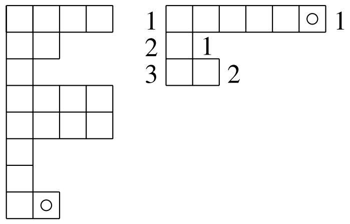
Now we evaluate riggings r_i according to equation (10):

$$\begin{aligned} r_1 &= \sum_{i=1}^{8-1} \min(2, \lambda_i) + E_{2,8} - 2\sum_{i=1}^{8} E_{2,i} \\ &= (2+2+1+2+2+1+1) + 1 - 2(0+0+1+1+1+0+1+1) \\ &= 2, \\ r_2 &= \sum_{i=1}^{5-1} \min(1, \lambda_i) + E_{1,5} - 2\sum_{i=1}^{5} E_{1,i} \end{aligned}$$

$$
\begin{aligned}
&= (1+1+1+1)+1-2(0+0+1+0+1)\\
&= 1,\\
r_3 &= \sum_{i=1}^{8-1}\min(6,\lambda_i)+E_{6,8}-2\sum_{i=1}^{8}E_{6,i}\\
&= (4+2+1+4+4+1+1)+2-2(0+0+1+2+3+0+1+2)\\
&= 1.
\end{aligned}
$$

Therefore we obtain $(\mu_1, r_1) = (2, 2)$, $(\mu_2, r_2) = (1, 1)$ and $(\mu_3, r_3) = (6, 1)$, which coincide with the following computation according to the original definition of ϕ. In the following, we use the Young diagrammatic expression for the rigged configurations, and we put riggings and vacancy numbers on the right and on the left of the corresponding rows of the configuration, respectively.



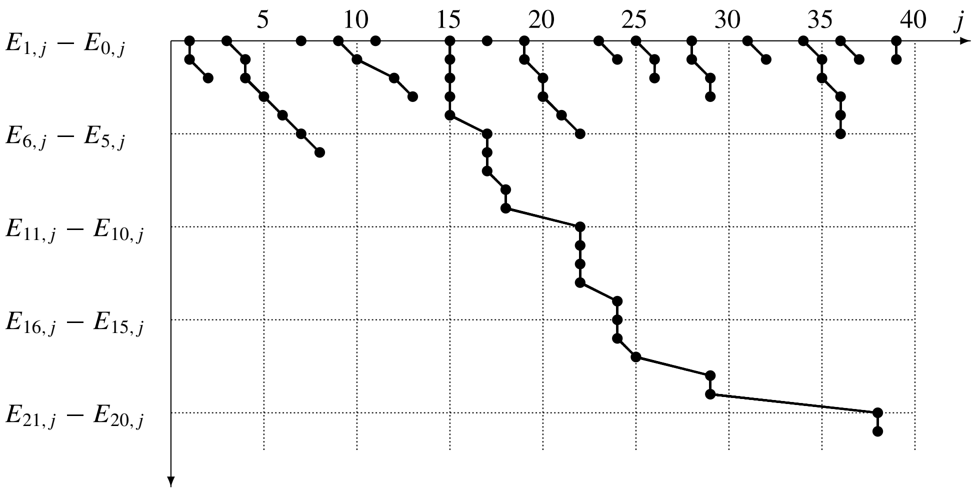


In the above diagrams, newly added boxes are indicated by circles '○'. The reader should compare the local energy distribution and the above box-adding procedure. Then one will observe that the local energy distribution and box addition on the μ part have close relationships. This relation will be established in lemma 4.2. In other words, the original combinatorial procedure for ϕ is embedded into rather automatic applications of the combinatorial R and energy functions.

Example 3.5. By using theorem 3.3, we can easily grasp the large-scale structure of combinatorial procedures of the KKR bijection from calculations of the combinatorial R and energy functions. In order to show the typical example, consider the following long path (length 40):

$$\boxed{122} \otimes \boxed{11112} \otimes \boxed{112} \otimes \boxed{22} \otimes \boxed{2} \otimes \boxed{12} \otimes \boxed{22} \otimes \boxed{111112} \otimes \boxed{2} \otimes \boxed{12} \otimes \boxed{2}$$
$$\otimes \boxed{2} \otimes \boxed{11112} \otimes \boxed{11} \otimes \boxed{122222} \otimes \boxed{1} \otimes \boxed{2222} \otimes \boxed{111122} \otimes \boxed{1122} \otimes \boxed{22} \otimes \boxed{2}$$
$$\otimes \boxed{122222} \otimes \boxed{12} \otimes \boxed{12222} \otimes \boxed{1122} \otimes \boxed{1122} \otimes \boxed{1} \otimes \boxed{122} \otimes \boxed{112222} \otimes \boxed{1} \otimes \boxed{2}$$
$$\otimes \boxed{1112} \otimes \boxed{1} \otimes \boxed{12} \otimes \boxed{122} \otimes \boxed{12222} \otimes \boxed{2} \otimes \boxed{1122} \otimes \boxed{122} \otimes \boxed{1}$$

Then, the local energy distribution takes the following form:



In the above table, letters 1 in the local energy distribution are represented by '•', and letters 0 are suppressed. By doing steps 2 and 3, we obtain classifications of letters 1. In the above table, letters 1 belonging to the same group are joined by thick lines. We see that there are 15 groups whose cardinalities are 3, 7, 1, 4, 1, 22, 1, 6, 2, 3, 4, 2, 6, 2, 2 from left to right, respectively.

By using equation (10), we obtain the unrestricted rigged configuration as follows: $(\mu_1, r_1) = (2, 13)$, $(\mu_2, r_2) = (2, 13)$, $(\mu_3, r_3) = (6, 15)$, $(\mu_4, r_4) = (2, 12)$, $(\mu_5, r_5) = (4, 14)$,

$(\mu_6, r_6) = (3, 12)$, $(\mu_7, r_7) = (2, 11)$, $(\mu_8, r_8) = (6, 5)$, $(\mu_9, r_9) = (1, 3)$, $(\mu_{10}, r_{10}) = (22, -17)$, $(\mu_{11}, r_{11}) = (1, 1)$, $(\mu_{12}, r_{12}) = (4, 1)$, $(\mu_{13}, r_{13}) = (1, 1)$, $(\mu_{14}, r_{14}) = (7, -3)$ and $(\mu_{15}, r_{15}) = (3, -2)$. The vacancy numbers for each row are $p_{22} = -15$, $p_7 = 15$, $p_6 = 19$, $p_4 = 21$, $p_3 = 18$, $p_2 = 14$ and $p_1 = 10$. Note that since the path in this example is not the highest weight, the resulting unrestricted rigged configuration has negative values of the riggings and vacancy numbers.

We have an alternative form of theorem 3.3.

Theorem 3.6. *In theorem 3.3, step 2 can be replaced by the following procedure (step 2′). The resulting groups are the same as those obtained in theorem 3.3 up to reordering in subscripts.*

(2′) Pick one of the lowest 1 *of the local energy distribution arbitrary, and denote it by* (μ_1, j_1)*. Starting from* (μ_1, j_1)*, choose one* 1 *from each row successively as follows. Assume that we have chosen* 1 *at* (l, k_l)*. Then* $(l-1, k_{l-1})$ *is the rightmost* 1 *among the part of row* $(l-1, 1), (l-1, 2), \ldots, (l-1, k_l)$*. Change all selected* 1 *into* 0*.*

Note that comparing both theorems 3.3 and 3.6, the resulting $(\lambda, (\mu, r))$ can be different in ordering of μ. However, this difference has no role in the KKR theory. By the same reason, an ambiguity in choosing the lowest 1 in step 2′ brings no important difference. The proof of theorem 3.6 will be given in the following section.

The formalism in theorem 3.6 is suitable for analysis of the periodic box-ball system. In particular, consider the case when there are more than one longest group in the local energy distribution. Choose any successive longest groups and apply step 2′ to these two groups. Then due to the non-crossing property of groups (proposition 4.4), we can concentrate on the region between these two groups and determine all groups between them ignoring the other part of the path.

Remark 3.7. Let us remark how the above formalism works for analysis of the periodic box-ball systems. We concentrate on the path b of the form $B_1^{\otimes L}$, where the number of $\boxed{2}$ is equal to or less than that of $\boxed{1}$. We define $v_l \in B_l$ by the relation $u_l \otimes b \simeq T_l(b) \otimes v_l$ with $T_l(b) \in B_1^{\otimes L}$. Then we have $v_l \otimes b \simeq \bar{T}_l(b) \otimes v_l$ with $\bar{T}_l(b) \in B_1^{\otimes L}$ (proposition 2.1 of [20]). $\bar{T}_l$'s are the time evolution operator of the periodic box-ball systems and $\bar{T}_1$ is simply the cyclic shift operator.

Consider the path $b^{\otimes N} = b \otimes \cdots \otimes b$. Then, from the property of v_l, we have $T_l(b^{\otimes N}) = T_l(b) \otimes \bar{T}_l(b) \otimes \cdots \otimes \bar{T}_l(b)$, i.e. we can embed the periodic box-ball system into the usual linear system with the operator T_l. Let us consider the local energy distribution for $b^{\otimes N}$. From the property $v_l \otimes b \simeq \bar{T}_l(b) \otimes v_l$, we see that under the right $(N-1)$ copies of b in $b^{\otimes N}$, we have $(N-1)$ copies of the same pattern of the local energy distribution.

Look at the local energy distribution below the rightmost b. In view of theorem 3.6 and the comments following it, a convenient way to find its structure is as follows. Instead of using u_l, we put v_l on the left of the path and draw the local energy distribution. In step 2′ of theorem 3.6, we choose the rightmost 1 from the $(l-1)$th row. If there is no such 1, we return to the right end of the $(l-1)$th row and find such 1.

In such a periodic extension of the local energy distribution, we can always find a boundary of two successive columns where none of the groups cross the boundary. By applying $\bar{T}_1$, we can move such a boundary to the left end of the path. We assume that b has already such a property. In our case, we can always do such a procedure, since by an appropriate choice of d, we can always make $\bar{T}_1^d(b)$ the highest weight (such d is not unique). Then this $\bar{T}_1^d(b)$ meet the condition (see lemma C.1 of [20] and lemma 4.2).

To summarize, by applying appropriate cyclic shifts, we can always make $b^{\otimes N}$ whose local energy distribution is N times repetition of the pattern for single b. On this property, we can apply the arguments of [21, 22] (combined with the tau function of [17]) to get the tau function in terms of the ultradiscrete Riemann theta function. More systematic treatment is given in [27].

Finally, we remark one thing without giving details (see section 3.3 of [27]). Recall that there is ambiguity in the choice of the cyclic shifts in the last paragraph. Let $\bar{T}_1^d(b)$ and $\bar{T}_1^{d'}(b)$ be two such possible choices (we assume $d' = 0$ for the sake of simplicity). Consider the local energy distribution for b. If the left d columns contain a group of cardinality l (or, in other words, if the difference between b and $\bar{T}_1^d(b)$ is a soliton of length l), then the riggings corresponding to b and $\bar{T}_1^d(b)$ differ by the operator σ_l called the slide (see section 3.2 of [20] for definition of slides). The slides σ_l are closely related to the period matrix of the tau functions of [21, 22] (see section 4 of [20]).

4. Proof of theorems 3.3 and 3.6

For the proof of theorem 3.3, we prepare some lemmas.

Lemma 4.1. *Let $(\lambda, (\mu, r))$ be the (unrestricted) rigged configuration corresponding to the path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$. Then we have (see (A.3) for definition of $Q_l^{(1)}$)*

$$E_l = Q_l^{(1)}. \tag{12}$$

Proof. We consider the path $b' := b \otimes \boxed{1}^{\otimes\Lambda}$, where $\Lambda \gg |\lambda|$. If b' is not highest, apply lemma A.2 and we can use the same argument which is given below. Since E_l is a conserved quantity on b', we have $E_l\big(T_\infty^{t_0}(b')\big) = E_l(b')$. We take t_0 large enough with the condition $\Lambda \geqslant t_0|\lambda|$ (the last inequality serves to assure that both $T_\infty^{t_0}(b')$ and b' contain the same number of letters 2). As we will see in the following, $T_\infty^{t_0}(b')$ has a simplified structure, so that we can evaluate $E_l\big(T_\infty^{t_0}(b')\big)$ explicitly.

Now we use proposition B.1. Since the actions of T_∞ cause linear evolution of riggings, we can assume the (unrestricted) rigged configuration corresponding to $T_\infty^{t_0}(b')$ as $(\lambda \cup (1^\Lambda), (\mu, \bar{r}))$. By the assumption $t_0 \gg 1$, these $\bar{r}$ have a simple property. Recall that in (B.4), if we apply T_∞ for one time, the rigging r_i corresponding to the row μ_i becomes $r_i + \mu_i$. Therefore, the riggings $\bar{r}_i$ and $\bar{r}_j$ corresponding to the rows μ_i and μ_j satisfy $\bar{r}_i \gg \bar{r}_j$ if $\mu_i > \mu_j$.

Using these observations, we determine the shape of $T_\infty^{t_0}(b')$ from $(\lambda \cup (1^\Lambda), (\mu, \bar{r}))$. By the assumption $t_0 \gg 1$, all letters 2 in $T_\infty^{t_0}(b')$ are contained in the $B_1^{\otimes\Lambda}$ part of the path. Therefore, corresponding to the row μ_i, there is a soliton of the form $\boxed{2}^{\otimes\mu_i}$. For example, in the following path,

$$\cdots \otimes \boxed{1} \otimes \boxed{2} \otimes \overbrace{\boxed{1} \otimes \boxed{1} \otimes \cdots \otimes \boxed{1} \otimes \boxed{1}}^{\gg 1} \otimes \boxed{2} \otimes \boxed{2}$$
$$\otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \cdots \tag{13}$$

there are one soliton of length 1 and two solitons of length 2. Since the riggings satisfy $\bar{r}_i \gg \bar{r}_j$ if $\mu_i > \mu_j$, the shorter solitons are located on the far left of longer solitons (see the above example).

Assume that there are solitons of the same length such as $\boxed{2}^{\otimes\mu_2} \otimes \boxed{1}^{\otimes\sigma} \otimes \boxed{2}^{\otimes\mu_1}$ $(\mu_1 = \mu_2)$. Then we show $\sigma \geqslant \mu_1 = \mu_2$. Let the riggings corresponding to the rows

μ_1 and μ_2 be r_1 and r_2, respectively. In order to minimize σ, we choose $r_1 = r_2$. Now we consider ϕ^{-1} on rows μ_1 and μ_2. Since we assume $t_0 \gg 1$, we do not need to consider the rows whose widths are different from μ_1. From $r_1 = r_2$, rows μ_1 and μ_2 become simultaneously singular, and we can choose one of them arbitrary. We remove μ_1 first. While removing boxes from μ_1 one by one, the shortened row μ_1 is always made singular, and the rows whose lengths are shorter than μ_1 are not singular. Therefore, we can remove the entire row μ_1 successively. After removing row μ_1, $Q^{(0)}_{\mu_2}$ decrease by μ_1 (note that the shape of the removed part of the quantum space is (1^{μ_1})) and $Q^{(1)}_{\mu_2}$ also decrease by μ_1 (because of the removal of μ_1). Since the vacancy number is defined by $Q^{(0)}_{\mu_2} - 2Q^{(1)}_{\mu_2}$, the vacancy number for the row μ_2 increases by μ_1 compared to that calculated before removing μ_1. Therefore, in order to make the row μ_2 singular again, we have to remove extra μ_1 boxes from the quantum space, without removing boxes of the μ part. Hence we have $\sigma \geqslant \mu_1$, as requested.

Now we are ready to evaluate $E_l\big(T^{t_0}_\infty(b')\big)$. From the definition of the combinatorial R, we have

$$\boxed{\overbrace{11\cdots1}^{l}} \otimes \boxed{2}^{\otimes m} \otimes \boxed{1}^{\otimes M} \simeq \boxed{1}^{\otimes \min(l,m)} \otimes \boxed{2}^{\otimes \max(m-l,0)} \otimes \boxed{\overbrace{11\cdots1}^{\max(l-m,0)}\ \overbrace{22\cdots2}^{\min(l,m)}} \otimes \boxed{1}^{\otimes M} \tag{14}$$

As we have seen, if there is a soliton of length m, there is always an interval longer than m, i.e. it has the form $\cdots \otimes \boxed{2}^{\otimes m} \otimes \boxed{1}^{\otimes M} \otimes \cdots$ with $m \leqslant M$. This makes $\boxed{11\cdots122\cdots2}$ of (14) into the form $\boxed{11\cdots1} = u_l$ when it comes to the left of the next (or right) soliton. Therefore, in order to evaluate $E_l\big(T^{t_0}_\infty(b')\big)$, we only have to consider a situation like (14). Noting the fact that the energy function, i.e. the unwinding number gains its value from the unwinding pair $\boxed{1\cdots} \otimes \boxed{2}$ appearing in the each tensor product (more precisely, E_l gains $\min(l,m)$ corresponding to the procedure (14)), the proof of lemma finishes. □

Combining the property of the combinatorial R with lemma 4.1, the relationship between the local energy distribution and the KKR bijection can be clarified as follows.

Lemma 4.2. *For the given path $b = b_1 \otimes \cdots \otimes b_i \otimes \cdots \otimes b_L$, draw local energy distribution. Within the ith column, denote the locations of* 1 *as $(j_1, i), (j_2, i), \ldots, (j_k, i)$ $(j_1 < j_2 < \cdots < j_k)$. Consider the calculation of $\phi(b) = (\lambda, (\mu, r))$. During the whole process of $\phi(b)$, when we create part of the (unrestricted) rigged configuration from b_i of b, we add boxes to columns $j_1, j_2, \ldots, j_k$ of μ in this order.*

Proof. Denoting $b_i = (x_1, x_2)$, let us define $b_{i,s} = (0, s)$ for $s \leqslant x_2$. Consider the path $b_s = b_1 \otimes b_2 \otimes \cdots \otimes b_{i,s}$ and draw local energy distribution for this b_s. In the local energy distribution, from the first column to the $(i-1)$th column are identical to those in the case for the original b. On the other hand, from the $(i+1)$th column to the Lth column, local energies are all 0. These are obvious from the construction of b_s.

Now consider the ith column of the local energy distribution for b_s. Then we show that the ith column is obtained from that corresponding to the original b by making s letters 1 from the top as it is, and letting all other letters be 0. This follows from the property of the combinatorial R, that is, the order of making pairs of dots does not affect the resulting image of the combinatorial R. In fact, in calculating $E_{l,i}$, we have $u_l^{(i-1)} \otimes b_{i,s}$. Compare this with $u_l^{(i-1)} \otimes b_i$. Then we can make a pair of dots in $u_l^{(i-1)} \otimes b_i$ such that first we join unwinding pairs and then we join winding pairs. Note that all letters 1 contained in b_i here cannot contribute as unwinding pairs. Therefore, we see that when we consider $u_l^{(i-1)} \otimes b_{i,s}$, $E_{l,i}$ $(l = 1, 2, \ldots)$ are the same with the $u_l^{(i-1)} \otimes b_i$ case up to the first s unwinding pairs, and we do not have the rest of the unwinding pairs. This verifies the assertion for the ith column of the local energy distribution for b_s.

Compare the local energy distribution for $b_1 \otimes \cdots \otimes b_{s-1}$ and $b_1 \otimes \cdots \otimes b_s$. Then, from the above observation, there is extra one 1 at (j_s, i). Now we apply the relation $E_l = Q_l^{(1)}$ (lemma 4.1) to both $b_1 \otimes \cdots \otimes b_{s-1}$ and $b_1 \otimes \cdots \otimes b_s$. Then we see that the letter 1 at (j_s, i) corresponds to the addition of one box at column j_s of μ of $\phi(b_{s-1})$. Since the μ part of $\phi(b)$ is obtained by adding boxes to μ recursively as $\phi(b_1), \phi(b_2), \ldots, \phi(b_{i,1}), \phi(b_{i,2}), \ldots,$ this gives the proof of lemma. □

Lemma 4.3. *Let $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}$ be an arbitrary path. b can be the highest weight or non-highest weight. Set $N = E_1(b)$. We determine the numbers $\mu_1, \mu_2, \ldots, \mu_N$ by the following procedure from steps 1 to 3.*

(1) *Draw a table containing $(E_{l,j} - E_{l-1,j} = 0, 1)$ at the position (l, j), i.e. at the lth row and the jth column. We call this table local energy distribution.*
(2) *Starting from the rightmost 1 in the $l =$ 1st row, pick the nearest 1 from each successive row. The 1 in the $(l+1)$th row must be weakly right of the 1 selected in the lth row. If there is no such 1 in the $(l+1)$th row, the position of the lastly picked 1 is called (μ_1, j_1). Change all selected 1 into 0.*
(3) *Repeat step 2 for $(N - 1)$ times to further determine $(\mu_2, j_2), \ldots, (\mu_N, j_N)$, thereby making all 1 into 0. Then μ coincides with μ of the (unrestricted) rigged configuration $\phi(b) = (\lambda, (\mu, r))$.*

Proof. We first interpret step 2 in terms of the original combinatorial procedure ϕ. In step 2, we choose the rightmost 1 of the first row of local energy distribution. From lemma 4.2, this 1 corresponds to the leftmost box of the lastly created row of μ. Suppose we choose letters 1 up to the lth row according to step 2. Next we choose 1 in the $(l+1)$th row, whose position is weakly right of 1 in the lth row. Since the lastly created row grows by adding boxes one by one during the procedure ϕ, this means that these two 1 at lth and $(l+1)$th rows of the local energy distribution belong to the same row (lastly created row) of μ. Note that if there are more than one row with the same length l, we can always add a box to the lastly created row, since it has maximal riggings among the rows with the same length. This follows from the fact that after adding a box at the lth column of the lastly created row of μ, the row is made to be singular, i.e. the row is assigned the maximal possible rigging for the row with length l. To summarize, step 2 ensures us to identify all 1 in the local energy distribution which correspond to the lastly created row of μ.

In step 3, we do the same procedure for the rest of 1 in the local energy distribution. Since we omit all letters 1 which are already identified with some rows of μ, we can always use step 2 to determine the next row. Therefore, step 3 ensures us to identify all 1 in the local energy distribution with the rows of μ. □

For the proof of theorem 3.3, we show that the groups of letters 1 obtained in lemma 4.3 have a simplified structure.

Proposition 4.4. *The groups of letters 1 obtained in steps 2 and 3 of lemma 4.3 have no crossing with each other.*

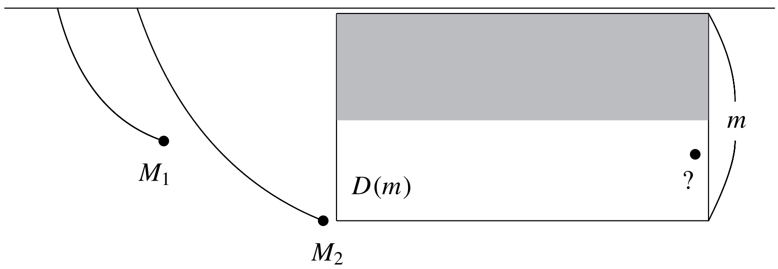
Proof. The proof is divided into six steps. In step 1, we analyze the geometric property of the crossing of groups. In step 2, we make the assumptions about crossing. In step 3, we analyze the behavior of corigging (= vacancy number − rigging) under the operation of ϕ. Then we introduce a convenient graphical method to analyze the coriggings by using '○' and '•'.

Here '•' represents the letters 1 contained in the local energy distribution and '∘' represents the change of the quantum space induced by letters 1 contained in the path. In step 4, we derive the relations from the assumption made in step 2. In step 5, we consider how to minimize the number of '∘' for a given pattern of '•' of the local energy distribution. Finally, in step 6, we show that the combination of the relations derived in step 4 and arguments in step 5 leads to the contradiction; hence it completes the proof.

Step 1. Consider the path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$ and calculate the local energy distribution corresponding to $b_{[k]} = b_1 \otimes b_2 \otimes \cdots \otimes b_k$ $(k \leqslant L)$. Recall that in lemma 4.2, we have shown that patterns of the local energy distribution represent the combinatorial procedures of the KKR bijection ϕ. Suppose that there are two groups of letters 1 whose cardinalities are m_1 and m_2, respectively, below $b_{[k]}$. We name these two groups as M_1 and M_2, respectively. Here we take that the top end of the group M_1 is located to the left of that of the group M_2.

From the geometric property of crossing, we show that, by an appropriate choice of k, we can assume $m_1 < m_2$ without crossing beneath $b_{[k]}$. Assume that there is a crossing between the lth row and the $(l+1)$th row, whereas there is no crossing above it. Denote the elements of M_1 and M_2 at the lth row by (l, m_{1-}) and (l, m_{2-}) where $m_{1-} < m_{2-}$, and the elements of m_1 and m_2 at the $(l+1)$th row by $(l+1, m_{1+})$ and $(l+1, m_{2+})$ where $m_{2+} < m_{1+}$, respectively. From the procedure given in step 2 of lemma 4.3, we have $m_{2-} \leqslant m_{2+}$. Now we choose k such that $m_{2+} \leqslant k < m_{1+}$ is satisfied. Since k satisfies $m_{1-} < m_{2-} \leqslant m_{2+} \leqslant k$, cardinality of the group M_1 beneath $b_{[k]}$ is l, whereas that of the group M_2 is equal to or greater than $l+1$, which gives the claim.

Step 2. We keep the notation like m_{i+}, as before; therefore, we have $m_1 < m_2$ beneath $b_{[m_{1+}-1]}$. Again, we assume that the crossing of M_1 and M_2 occurs beneath $b_{[m_{1+}]}$, and also that there is no crossing beneath $b_{[m_{1+}-1]}$. We denote the cardinalities of the groups M_1 and M_2 under $b_{[m_{1+}-1]}$ by m_1 and m_2, respectively. At the end of the proof, we will show that the existence of the crossing leads to contradiction. We consider the crossing of two groups, since this is the fundamental situation. The general case involving more than two crossings follows from this fundamental case. The situation here is depicted in the following diagram:



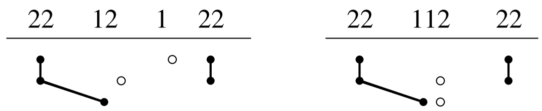
Here, letters 1 in the local energy distribution are represented by '•' and all letters 0 are suppressed. Note that we have introduced the domain $D(m)$ on the right of the bottom point of M_2, occupying from the first row to the mth row. Since we consider the crossing caused by two groups, we can assume that groups contained in $D(m_2)$ are, in fact, contained in $D(m_1 - 1)$ (in the above diagram, it is indicated by the gray rectangle). If we can put '•' indicated by '?' in the above diagram, then the crossing of the groups M_1 and M_2 occurs.

In the following, we first treat the case that M_2 and other groups on the right of M_2 are well separated. This means that the other groups on the right of M_2 are located on the right of the bottom points of M_2, and between the top and the bottom point of M_2, there is no '∘'

on the right of M_2 (see step 3 for meaning of '$\circ$'). This assumption is only for the sake of simplicity, and the general case will be mentioned at the end of the proof.

Step 3. We summarize the basic properties of the vacancy numbers (or, at the same time, those of the coriggings). Recall the definition of the vacancy numbers $p_j = Q_j^{(0)} - 2Q_j^{(1)}$ corresponding to the pair (λ, μ). Consider the box-adding procedure of ϕ. If we add boxes to $\lambda|_{\leqslant j}$ and $\mu|_{\leqslant j}$ simultaneously, then the vacancy number p_j decreases by 1. In contrast, if we add a box to $\lambda|_{\leqslant j}$ and do not add a box to $\mu|_{\leqslant j}$, then the vacancy number increases by 1. Note that if we do not add a box to both $\lambda|_{\leqslant j}$ and $\mu|_{\leqslant j}$, then the vacancy number does not change. Also recall that the procedure ϕ only refers to corigging.

In order to analyze the above change of coriggings induced by the box-adding procedure of ϕ, it is convenient to supplement the local energy distribution with the information of change of the quantum space corresponding to letters 1 contained in the path. In the local energy distribution, we replace letters 1 by '$\bullet$' and suppress all letters 0. We join '$\bullet$' belonging to the same group by thick lines. Then, corresponding to the letters 1 contained in b_s, we put '$\circ$' on the right of '$\bullet$' corresponding to the letters 2 contained in b_s. The row coordinates of '$\circ$' are taken to be the same as the column coordinate of the added box of the quantum space induced by the corresponding letters 1. Here we give examples of such a diagram for $\boxed{22} \otimes \boxed{12} \otimes \boxed{1} \otimes \boxed{22}$ and $\boxed{22} \otimes \boxed{112} \otimes \boxed{22}$, respectively.



In this diagram, the KKR map ϕ proceeds from the left to right, and within the same column (distinguishing columns of '$\bullet$' and '$\circ$'), it proceeds from the top to bottom. In the above examples, the left group containing three '$\bullet$' became singular (i.e. corigging = 0) after the bottom '$\bullet$' is added. Then the two '$\circ$' increase the corigging by 2; therefore the right group containing two '$\bullet$' stays independently from the left group. This kind of analysis of change of the coriggings is a prototype of the arguments given in steps 5 and 6. Note that along each group of '$\bullet$', the notion of left and right of the group is well defined. Let us remark the convenient method to determine locations of '$\circ$'. Given an element $\boxed{1 \cdots 12 \cdots 2}$, we reverse the orderings of numbers as $\boxed{2 \cdots 21 \cdots 1}$. Choose the specific letter 1 and denote by p the number of letters 1 and 2 on the left of it. Then, corresponding to the chosen 1, we put '$\circ$' on the $(p + 1)$th row on the local energy distribution.

Step 4. Assume that we are going to add the box corresponding to $(m_1 + 1, m_{1+})$ of the local energy distribution. In order to add a box corresponding to $(m_1 + 1, m_{1+})$, or in other words, in order to make crossing, the row of μ corresponding to M_2 cannot be singular when we add the box corresponding to $(m_1 + 1, m_{1+})$. This follows from the assumption $m_1 < m_2$ and the fact that we add a box to the longest possible singular row in the procedure ϕ. Also, the row of μ corresponding to M_1 has to be singular in order to add a box corresponding to $(m_1 + 1, m_{1+})$. We consider the implications of these two conditions.

Recall that the row of μ corresponding to the group M_2 is singular when the bottom '$\bullet$' is added to the end of M_2. On the other hand, we have to make M_2 non-singular as we have seen in the above. This means

$$(\text{number of '}\circ\text{' within } D(m_2)) > (\text{number of '}\bullet\text{' within } D(m_2)). \tag{15}$$

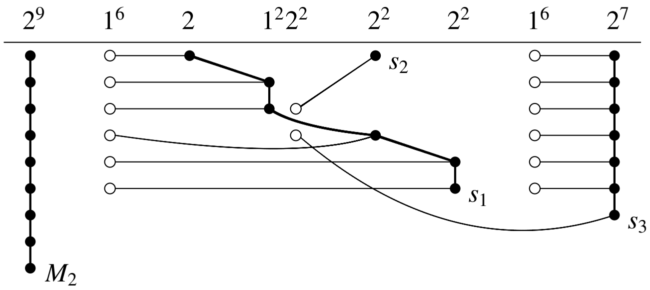
On the other hand, in order to make M_1 singular, we have

$$(\text{number of '}\circ\text{' within } D(m_1)) \leqslant (\text{number of '}\bullet\text{' within } D(m_1)). \tag{16}$$

Note that all '$\bullet$' are contained in $D(m_1 - 1)$. From these two restrictions, we see that there are at least one '$\circ$' in $D(m_2)\backslash D(m_1)$.

Step 5. If we are given the pattern of '$\bullet$', there remains various possibilities about the pattern of '$\circ$'. Now we are going to consider the patterns of '$\circ$' that minimize the number of '$\circ$'. To say the result at first, we see that we need '$\circ$' as much as '$\bullet$'; therefore, in order to meet condition (16), we have to minimize the number of '$\circ$'.

Suppose there are two groups in the local energy distribution: the group s_1 on the left and s_2 on the right. Let the top '$\bullet$' be located at columns k_1 and k_2. Then, in order to make s_1 and s_2 as separated groups, we need $\min(s_1, s_2)$ '$\circ$' within the region between (or surrounded by) s_1 and column k_2. To make the situation transparent, we consider the concrete path $\boxed{222222222} \otimes \boxed{111111} \otimes \boxed{2} \otimes \boxed{1122} \otimes \boxed{22} \otimes \boxed{22} \otimes \boxed{111111} \otimes \boxed{2222222}$. Then the corresponding diagram is as follows:



We see that there are four groups, labeled by M_2, s_1, s_2, s_3 respectively from left to right. These groups are indicated by thick lines. We can analyze the situation as follows (as for the method for analysis of change of the coriggings, see the latter part of step 3).

(1) In order to make M_2 and s_1 separated, we need six '$\circ$' between M_2 and s_1. The precise meaning of 'between M_2 and s_1', etc, is given after the example. In this example, they are supplied by six '$\circ$' coming from the left $\boxed{111111}$. This situation is indicated by thin lines, which join the corresponding '$\circ$' and '$\bullet$'. Of course, there is ambiguity in the way of joining '$\circ$' and '$\bullet$'. However, this ambiguity brings no important effect; hence it is neglected. For example, we can join top five '$\circ$' coming from the left $\boxed{111111}$ and the bottom '$\circ$' coming from $\boxed{1122}$ with six '$\bullet$' of s_1. In such a case, the bottom '$\circ$' coming from the left $\boxed{111111}$ should be connected with the bottom '$\bullet$' of s_3.

(2) In order to make s_1 and s_2 separated, we need one '$\circ$' between s_1 and s_2. In this example, it is supplied by one '$\circ$' coming from $\boxed{1122}$.

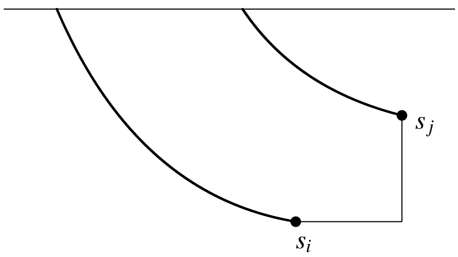
(3) In order to make s_2 and s_3 separated, we need one '$\circ$' between s_2 and s_3. In this example, it is supplied by the top '$\circ$' coming from the right $\boxed{111111}$.

(4) In order to make s_1 and s_3 separated, we need five '$\circ$' between s_1 and s_3. In this example, it is supplied by the bottom five '$\circ$' coming from the right $\boxed{111111}$.

(5) In order to make M_2 and s_3 separated, we need one '$\circ$' between M_2 and s_3. In this example, it is supplied by one '$\circ$' coming from $\boxed{1122}$.

Let us remark that if we move one 1 of the second term of the above path to the seventh term, i.e., $\boxed{222222222} \otimes \boxed{11111} \otimes \boxed{2} \otimes \boxed{1122} \otimes \boxed{22} \otimes \boxed{22} \otimes \boxed{1111111} \otimes \boxed{2222222}$ has exactly the same pattern of '$\bullet$' as the above example.

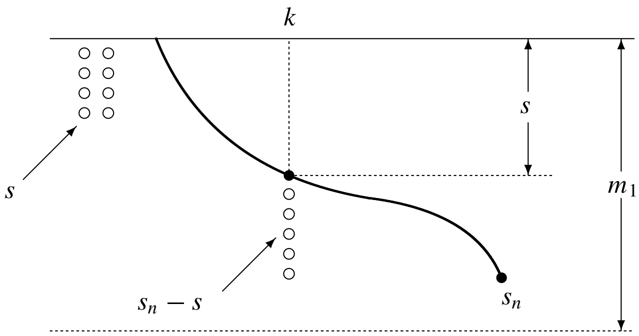
From this example, we can infer the general case. Assume that there are groups $s_1, \ldots, s_n$ (from left to right) on the right of M_2. We define the region between s_i and s_j $(i < j)$ as the region surrounded by s_i, the first row and the row containing the bottom point of s_i, s_j and the column containing the bottom point of s_j (except s_i, s_j and the column containing the bottom point of s_j, see the following diagram).



We choose an ordered subsequence $s_{j_1}, \ldots, s_{j_p}$ of groups $s_1, \ldots, s_n$ such that it is the longest subsequence which satisfies $s_{j_1} > \cdots > s_{j_p} > s_{n-1}$. Between s_{n-1} and s_n, we need at least $\min(s_n, s_{n-1})$ '∘' in order to make s_{n-1} and s_n separate. If $s_n > s_{n-1}$, then we compare s_n and s_{j_p}. Then we need at least $\min(s_n - s_{n-1}, s_{j_p} - s_{n-1})$ '∘' between s_{j_p} and s_n in order to make s_{j_p} and s_n separate. We continue this process and, in conclusion, we need '∘' as much as '•', compared within the right of M_2.

Step 6. Based on the ground of the arguments given in step 5, we derive the contradiction against the statement 'there are at least one '∘' in $D(m_2)\backslash D(m_1)$'.

Again, assume that there are groups $s_1, \ldots, s_n$ on the right of M_2. Denote the column coordinate of the top '•' of s_j by k_j and that of the bottom of s_j by k'_j. In order to meet condition (16), we choose the pattern of '∘' which minimizes the number of '∘'. From observations made in step 5, we have at most s_n '∘' between s_{n-1} and s_n. If all '∘' are located between the k_{n-1}th column and the k_nth column, then the number of '∘' (i.e. s_n) is too short to make '∘' appear in $D(m_2)\backslash D(m_1)$. Let us analyze the case when some of s_n '∘' appear between the k_nth column and the k'_nth column. To be specific, take some k between k_n and k'_n, and write the row of the lowest '•' of the column k belonging to the group s_n by s. In order to make the upper s '•' of s_n separated from s_{n-1}, it needs at least s '∘' on the left of the kth column. See the following schematic diagram:



Thus, we have at most $s_n - s$ '∘' below the sth row of the kth column. If we attach a column of $s_n - s$ '∘' to the kth row, it has to begin from the $(s + 1)$th row. Therefore, it is also too short to make '∘' appear in $D(m_2)\backslash D(m_1)$. Similarly, between s_{n-2} and s_{n-1}, we have at most $\max(s_n, s_{n-1})$, again too short to make '∘' appear in $D(m_2)\backslash D(m_1)$. Continuing in this way, we see that no '∘' appear in $D(m_2)\backslash D(m_1)$, which gives contradiction.

As we have claimed at the end of step 2, so far we are dealing only with the case that M_2 and the other groups on the right of it are well separated. However, we can treat the general case by similar arguments. First, by applying the same argument of step 5, we can show that we need '○' as much as '•' within the region on the right of the group M_2. Then, by applying the same argument in step 6, all '○' on the right of M_2 are included in the first $m_1 - 1$ rows of the local energy distribution. Therefore, in order to '○' appear in $D(m_2)\backslash D(m_1)$, we have to add at least one '○' within the first m_1 rows. This makes M_1 non-singular; hence, the crossing does not occur in this case.

Hence, we complete the proof of proposition. □

Proof of theorem 3.3. From proposition 4.4, we can remove the procedure to find 'the nearest 1' from step 2 of lemma 4.3. This gives the proof of steps 1 to 3 of theorem 3.3.

Finally, we clarify the meaning of step 4. In step 3 of lemma 4.3, we determined (μ_k, j_k), which corresponds to the rightmost box of row μ_k of μ. Since the row μ_k is not lengthened in calculation of $\phi(b)$ after b_{j_k}, the rigging of row μ_k is equal to the vacancy number at the time when we add the rightmost box to μ_k. At this moment, the quantum space takes the form

$$\left(\lambda_1, \lambda_2, \ldots, \lambda_{j_k-1}, E_{\mu_k, j_k}\right). \tag{17}$$

The meaning of the last E_{μ_k,j_k} is as follows. E_{μ_k,j_k} counts all letters 1 contained in the first μ_k rows of the j_kth column of local energy distribution. This means that, from lemma 4.2, we added E_{μ_k,j_k} boxes to μ before the rightmost box of the row μ_k is added (while considering b_{j_k}). In the procedure ϕ, we use letters 2 of b_{j_k} first and then use the rest of letters 1 of b_{j_k}. Since letters 2 in b_{j_k} mean simultaneous addition of the box to the quantum space and μ, we can conclude that the quantum space has the row with length $E_{\mu_k,j_k} (\leqslant \mu_k)$. From this shape of the quantum space, we have

$$Q^{(0)}_{\mu_k} = \sum_{i=1}^{j_k-1} \min(\mu_k, \lambda_i) + E_{\mu_k,j_k}. \tag{18}$$

From lemma 4.1 applying to the path $b_1 \otimes b_2 \otimes \cdots \otimes b_{j_k}$, we deduce the following:

$$Q^{(1)}_{\mu_k} = \sum_{i=1}^{j_k} E_{\mu_k,i}. \tag{19}$$

Hence, we obtain the formula in step 4 and complete the proof of theorem 3.3. □

Proof of theorem 3.6. This immediately follows from the non-crossing property of proposition 4.4. In theorem 3.3, we determine groups from right to left. More precisely, after determining one group, all letters 1 belonging to the group are made to be 0, and we determine the rightmost group again. However, if we start from the bottom point of the longest group, the elimination made in the right has no effect, and we can determine the same group by virtue of the non-crossing property. By doing these procedures from longer groups to shorter groups, we obtain the same groups as in the case of theorem 3.3. □

5. Summary

In this paper, we consider crystal interpretation of the KKR map ϕ from paths to rigged configurations. In section 3, we introduce a table called the local energy distribution. The entries of the table are differences of the energy functions, and we show in theorem 3.3 that this table has complete information about ϕ so that we can read off the rigged configuration from it. As we see in lemma 4.2, this table can be viewed as giving crystal interpretation of the combinatorial procedures appearing in the original definition of ϕ.

As we see in proposition 4.4, our formalism has a simple property. This enables us to reformulate theorem 3.3 as described in theorem 3.6. The latter formalism is particularly important when we consider the inverse scattering formalism for the periodic box-ball systems. As we see in remark 3.7, the advantage of our formalism, compared with the formalism given in [20], is that we can treat states of the periodic box-ball system directly without sending them to linear semi-infinite systems.

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Appendix A. Kerov–Kirillov–Reshetikhin bijection

In this section, we prepare notations and basic properties corresponding to the Kerov–Kirillov–Reshetikhin (KKR) bijection. As for the definitions of the rigged configurations as well as the combinatorial procedure of the bijection, we refer to section 2 of [16] (ϕ there should be read as ϕ^{-1} here) or appendix A of [20], and we only prepare necessary notations.

Assume that we have given a highest weight path b:

$$b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}. \tag{A.1}$$

Then we have one-to-one correspondence ϕ between b and the rigged configuration:

$$\phi : b \longrightarrow \mathrm{RC} = \left((\lambda_i)_{i=1}^{L}, (\mu_i, r_i)_{i=1}^{N}\right). \tag{A.2}$$

Here, $(\mu_i)_i \in \mathbb{Z}_{\geqslant 0}^N$ is called configuration and we depict $(\lambda_i)_i$ and $(\mu_i)_i$ by the Young diagrammatic expression whose rows are given by λ_i and μ_i, respectively. Integers r_i are called riggings and we associate them with the corresponding row μ_i. In the KKR bijection, orderings within integer sequences $(\lambda_i)_i$ or $(\mu_i, r_i)_i$ do not make any differences. On these rigged configurations, we use the symbols $Q_j^{(a)}$ $(a = 0, 1)$ defined by

$$Q_j^{(0)} := \sum_{k=1}^{L} \min(j, \lambda_k), \qquad Q_j^{(1)} := \sum_{k=1}^{N} \min(j, \mu_k). \tag{A.3}$$

The vacancy number p_j for length j rows of μ is then defined by

$$p_j := Q_j^{(0)} - 2Q_j^{(1)}. \tag{A.4}$$

If row μ_i has the property $p_{\mu_i} = r_i$, then the row is called singular. For the highest weight paths, the corresponding rigged configurations are known to satisfy $0 \leqslant r_i \leqslant p_{\mu_i}$. The quantity $p_{\mu_i} - r_i$ is sometimes called corigging.

One of the most important properties of ϕ or ϕ^{-1} is that if we consider isomorphic paths $b \simeq b'$, then the corresponding rigged configuration is the same (lemma 8.5 of [3]). We express this property in terms of the map ϕ^{-1} as follows.

Theorem A.1. *Take two successive rows from the quantum space λ of the rigged configuration to be arbitrary, and denote them by λ_a and λ_b respectively. When we remove λ_a at first and next λ_b by the KKR map ϕ^{-1}, then we obtain two tableaux, which we denote by a_1 and b_1, respectively. Next, in contrast, we first remove λ_b and then λ_a (keeping the order of other removal invariant) and we get b_2 and a_2. Then we have*

$$b_1 \otimes a_1 \simeq a_2 \otimes b_2, \tag{A.5}$$

under the isomorphism of $\mathfrak{sl}_2$ combinatorial R matrix.

We remark that there is an extension of ϕ which covers all non-highest weight elements as well. Let b be an arbitrary element of arbitrary tensor products of crystals; $b \in B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_L}$. In particular, b can be the non-highest weight element. Then we can apply the same combinatorial procedure for ϕ and obtain $\phi(b)$ as an extension of the rigged configurations. Following [28, 29], we call such $\phi(b)$ *unrestricted rigged configuration*. Let us denote $\phi(b) = (\lambda, (\mu, r))$. Then, from definition of ϕ, we see that $|\lambda|$ represents the number of all letters 1 and 2 contained in the path b, whereas $|\mu|$ represents the number of letters 2 contained in b. Note, in particular, that $|\lambda| \geqslant |\mu|$ holds for unrestricted rigged configurations. These unrestricted rigged configurations contain the rigged configurations as the special case.

Let b be a non-highest weight element as above. Consider the following modification of b:

$$b' := \boxed{1}^{\otimes \Lambda} \otimes b, \tag{A.6}$$

where Λ is an integer satisfying $\Lambda \geqslant \lambda_1 + \cdots + \lambda_L$. Then b' is highest weight. Under these notations, we have the following.

Lemma A.2. *Let the unrestricted rigged configuration corresponding to b be*

$$\left((\lambda_i)_{i=1}^{L}, (\mu_j, r_j)_{j=1}^{N}\right). \tag{A.7}$$

Then the rigged configuration corresponding to the highest path b' is given by

$$\left((\lambda_i)_{i=1}^{L} \cup (1^{\Lambda}), (\mu_j, r_j + \Lambda)_{j=1}^{N}\right). \tag{A.8}$$

Proof. Let the vacancy number of row μ_j of the pair (λ, μ) of (A.7) be p_{μ_j}. Then the vacancy number of row μ_j of (A.8) is equal to $p_{\mu_j} + \Lambda$, because of the addition of (1^{Λ}) on λ. Now we apply ϕ^{-1} on (A.8). From $\lambda \cup (1^{\Lambda})$ of the quantum space, we remove λ first and next remove (1^{Λ}). Recall that in the combinatorial procedure of ϕ^{-1}, we only refer to corigging, and it does not refer to actual values of the riggings. Therefore, when we remove λ from the quantum space of (A.8), we obtain b as the corresponding part of the image. Then, the remaining rigged configuration has the quantum space (1^{Λ}) without the μ part. On this rigged configuration, the map ϕ^{-1} becomes trivial and b' is obtained as the image corresponding to (A.8). □

Appendix B. Operators T_l

In this section, we introduce the operators T_l which are defined by the combinatorial R. T_l's serve as the time evolution operators of the box-ball systems [25]. Let u_l be a highest weight element of B_l, i.e. in a tableau representation, it is $u_l = \boxed{\overbrace{11\cdots 1}^{l}}$. We consider the path

$$b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}. \tag{B.1}$$

Then its time evolution $T_l(b)$ $(l \in \mathbb{Z}_{>0})$ is defined by successively sending u_l to the right of b under the isomorphism of the combinatorial R as follows:

$$\begin{aligned} u_l \otimes b &= u_l \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_L \\ &\overset{R}{\simeq} b_1' \otimes u_l^{(1)} \otimes b_2 \otimes \cdots \otimes b_L \\ &\overset{R}{\simeq} b_1' \otimes b_2' \otimes u_l^{(2)} \otimes \cdots \otimes b_L \\ &\overset{R}{\simeq} \cdots\cdots \\ &\overset{R}{\simeq} b_1' \otimes b_2' \otimes \cdots \otimes b_L' \otimes u_l^{(L)} \\ &=: T_l(b) \otimes u_l^{(L)}. \end{aligned} \tag{B.2}$$

According to proposition 2.6 of [13], operators T_l on highest paths can be linearized by the KKR bijection. Since, in the main text, we use the similar property for the general case including non-highest paths, we include here a proof for a generalized version.

Proposition B.1

(1) *Consider the path b of the form (B.1). Here b can be a non-highest weight element. Define* $b' = b \otimes \boxed{1}^{\otimes\Lambda}$*, where the integer* Λ *satisfies* $\Lambda > \lambda_1 + \lambda_2 + \cdots + \lambda_L$*. Then, we have* $u_l \otimes b' \simeq T_l(b') \otimes u_l$.

(2) *Denote the (unrestricted) rigged configuration corresponding to* b' *as*

$$b' \xrightarrow{\text{KKR}} \left(\lambda \cup (1^\Lambda), (\mu_j, r_j)_{j=1}^N\right). \tag{B.3}$$

Then, corresponding to $T_l(b')$*, we have*

$$T_l(b') \xrightarrow{\text{KKR}} \left(\lambda \cup (1^\Lambda), (\mu_j, r_j + \min(\mu_j, l))_{j=1}^N\right). \tag{B.4}$$

Proof

(1) Consider the elements $u_l^{(i)}$ defined in (B.2). In our case, $u_l^{(L)}$ contains letters 2 for at most $\lambda_1 + \lambda_2 + \cdots + \lambda_L$ times. Then, by calculating combinatorial R along (B.2) with $u_l^{(L)}$ and $\boxed{1}^{\otimes\Lambda}$, we see that $u_l^{(L+\Lambda)} = u_l$.

(2) Consider the following rigged configuration:

$$\left(\lambda \cup (1^\Lambda) \cup (l), (\mu_j, r_j + \min(\mu_j, l))_{j=1}^N\right), \tag{B.5}$$

i.e. we added a row with width l to the quantum space. Compare the coriggings of (B.5) and (B.3). Recall that the vacancy number is defined by $Q_{\mu_j}^{(0)} - 2Q_{\mu_j}^{(1)}$. As for $Q_{\mu_j}^{(1)}$, both (B.5) and (B.3) give the same value, since we have μ in both second terms. In contrast, $Q_{\mu_j}^{(0)}$ for (B.5) is greater than that for (B.3) by $\min(\mu_j, l)$, since we have the extra row of width l in the quantum space of (B.5). In (B.5), riggings are increased by value $\min(\mu_j, l)$; therefore, we conclude that the coriggings for both (B.5) and (B.3) coincide.

Now we apply ϕ^{-1} to (B.5) in two different ways. First, we remove $\lambda \cup (1^\Lambda)$ from the quantum space of (B.5) (order of removal is the same as ϕ^{-1} on (B.3) to obtain b'). Since the coriggings for both μ_j of (B.5) and (B.3) coincide, we obtain b' as the corresponding part of the image. Then we are left with the rigged configuration $(l, (\emptyset, \emptyset))$, which yields u_l. Therefore, we obtain $u_l \otimes b'$ as the image.

Next, we apply ϕ^{-1} on (B.5) in a different way. In this case, we remove the row l of the quantum space as the first step. Note that in the (unrestricted) rigged configuration $(\lambda, (\mu, r))$ corresponding to the path b, all riggings r_j are smaller than the corresponding vacancy numbers. By definition of Λ, we have $\Lambda > \min(\mu_j, l)$ for all j (recall that from the definition of the unrestricted rigged configuration, we always have $\lambda_1 + \cdots + \lambda_L \geqslant \mu_1 + \cdots + \mu_N$). As a result, if we remove the row l from the quantum space of (B.5), rows μ_j do not become singular even if the riggings are increased as $r_j + \min(\mu_j, l)$, since the vacancy numbers are also increased by Λ by the addition of (1^Λ). Thus, we obtain u_l as the corresponding part of the image. Then we are left with $\left(\lambda \cup (1^\Lambda), (\mu_j, r_j + \min(\mu_j, l))_{j=1}^N\right)$, whose corresponding path we denote by $\tilde{b}'$. In conclusion, we obtain $\tilde{b}' \otimes u_l$ as the image.

In the above two calculations of ϕ^{-1}, the only difference is the order of removing rows of the quantum space of (B.5). Therefore, we can apply theorem A.1 to get the isomorphism

$$u_l \otimes b' \overset{R}{\simeq} \tilde{b}' \otimes u_l. \tag{B.6}$$

If b' is the non-highest weight, we apply lemma A.2 and $u_l \otimes u_m \simeq u_m \otimes u_l$ and we can then use the same argument. From (1), this means $\tilde{b}' = T_l(b')$, which completes the proof. □

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